$\overline{\partial}$-equations, integrable deformations of quasiconformal mappings and Whitham hierarchy *

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Abstract

It is shown that the dispersionless scalar integrable hierarchies and, in general, the universal Whitham hierarchy are nothing but classes of integrable deformations of quasiconformal mappings on the plane. Examples of deformations of quasiconformal mappings associated with explicit solutions of the dispersionless KP hierarchy are presented.

Key words: Dispersionless hierarchies, quasiconformal mappings, $\overline{\partial}$-equations.
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1 Introduction

It is widely recognize now that the dispersionless or quasiclassical integrable equations and hierarchies form an important part of the theory of integrable systems (see e.g. [1]-[7]). They are the main ingredients of various approaches to different problems which arise in physics and applied mathematics, as in the theory of topological quantum fields [8, 9], hydrodynamics [1] or optical communications modelling [10]. Moreover, it was recently shown in [11, 12] that the dispersionless hierarchies provide an effective tool to study some classical problems of the theory of conformal maps.

A purpose of this letter is to demonstrate the existence of an intimate relation between dispersionless hierarchies and quasiconformal mappings on the complex plane. We show that the dispersionless scalar integrable hierarchies and, in general, the members of the universal Whitham hierarchy are nothing but infinite-dimensional rings of integrable deformations of quasiconformal mappings on the plane. The nonlinear $\bar{\partial}$-equation

$$S_{\bar{z}} = W\Big(z, \bar{z}, S_z\Big), \tag{1}$$

is the basic element of our analysis. Here $S(z, \bar{z}, \boldsymbol{t})$ is a complex-valued function depending an infinite set $\boldsymbol{t}$ of parameters (times), $S_{\bar{z}} := \frac{\partial S}{\partial \bar{z}},\ S_z := \frac{\partial S}{\partial z}$ and W is an appropriate function of $z, \bar{z}$ and S_z. Equation (1) is well-known in the theory of quasiconformal mappings. Namely, under some mild conditions its solutions define quasiconformal mappings on the plane (see [13]-[16]).

On the other hand, as it was shown in [17, 18] equation (1) is a quasiclassical limit of the standard nonlocal $\bar{\partial}$-problem arising in the $\bar{\partial}$-dressing method, which allows us to generate the dispersionless hierarchies. These two faces of equation (1) together with some available rigorous results about the Beltrami equation, lead to a natural derivation of the universal Whitham hierachy from equation (1) as well as an exact solution method for dispersionless hierarchies. To illustrate these facts we present examples of exact solutions of the dKP hierarchy or, equivalently, of explicit integrable deformations for the quasiconformal mappings determined by equation (1).

2 Quasiconformal mappings

Quasiconformal mappings are a natural and very rich extension of the concept of conformal mappings. For the sake of convenience we remind here some of

their basic properties (see e.g. [13, 14], [19]-[23]).

A sense-preserving homeomorphism $w = f(z, \bar{z})$ defined on a domain G of the complex plane $\mathbb{C}$ is called *quasiconformal* (qc) if

$$\frac{|f_z| + |f_{\bar{z}}|}{|f_z| - |f_{\bar{z}}|},$$

is bounded on G. Here f_z and $f_{\bar{z}}$ are assumed to be locally square-integrable generalized derivatives. Since the Jacobian of f is assumed to satisfy $J = |f_z|^2 - |f_{\bar{z}}|^2 > 0$ almost everywhere in G, the so-called *complex dilatation* of f $\mu_f := \frac{f_{\bar{z}}}{f_z}$ verifies $|\mu_f| < 1$ almost everywhere in G. Thus, a natural analytic characterization of qc-mappings can be given in terms of homeomorphic generalized solutions of the linear Beltrami equation

$$f_{\bar{z}} = \mu f_z, \tag{2}$$

where μ is any given measurable function $||\mu||_\infty < 1$ on G. Obviously, for $\mu \equiv 0$ we get into the class of conformal mappings. There is a qc-version of the Riemann's mapping theorem for conformal mappings. Namely, for any $||\mu||_\infty < 1$ on a simply-connected domain G there exists a homeomorphism $w = f(z, \bar{z})$ which solves (2) and maps G onto a given simply-connected domain Δ. A very simple example of qc-mapping is the affine mapping $w = az + b\bar{z} + c$ where a, b, c are complex constants with $|b| > |a|$. It maps every circle into a ellipse.

The properties of solutions of the Beltrami equation (2) are rather well studied (see e.g. [19]). Some of them are particularly important for our discussion. To present these results we need to introduce the operators [13]

$$Th(z) := \frac{1}{2\pi\imath} \iint_{\mathbb{C}\times\mathbb{C}} \frac{h(z')}{z' - z} \,\mathrm{d}\, z' \wedge \mathrm{d}\, \bar{z}', \quad \Pi(z) := \frac{\partial Th}{\partial z}(z),$$

where the integral is taken in the sense of the Cauchy principal value. Then one has [13]:

Lemma 1. *For any $p > 1$ the operator Π defines a bounded operator in $L^p(\mathbb{C})$ and for any $0 \le k < 1$ there exists $\delta > 0$ such that*

$$k||\Pi||_p < 1,$$

for all $|p - 2| < \delta$.

The next theorem summarizes the properties of qc-mappings that we need in the subsequent discussion [19, 21, 22].

Theorem 1. *Given a measurable function μ with compact support inside the circle $|z| < R$ and such that $||\mu||_\infty < k < 1$. Then, for any fixed exponent $p = p(k) > 2$ such that $k||\Pi||_p < 1$, it follows that*

1) *There is a function f_0 on $\mathbb{C}$ with distributional derivatives satisfying the Beltrami equation (2) such that*

$$f_0(z) = z + O(\frac{1}{z}), \quad z \to \infty, \tag{3}$$

with $f_{0,\bar{z}}$ and $f_{0,z} - 1$ being elements of $L^p(\mathbb{C})$. Any such f_0 is unique and determines a homeomorphism on $\mathbb{C}$ (the basic homeomorphism of (2)).

2) *Every solution of (2) on a domain G of $\mathbb{C}$ can be represented as*

$$f(z) = \Phi(f_0(z)), \tag{4}$$

where Φ is an arbitrary analytic function on the image domain $f_0(G)$ of G under the basic homeomorphism f_0.

Deformations of qc-mappings have been discussed by several authors. One can show [13, 20, 22, 23] that if $\mu(z, \bar{z}, \boldsymbol{t})$ depend analytically on one or several parameters $\boldsymbol{t}$, for fixed $z \in \mathbb{C}$, then the corresponding family of basic homeomorphisms $f_0(z, \bar{z}, \boldsymbol{t})$also depend analytically on $\boldsymbol{t}$. However, it seems that apart from concrete examples, as the isotropy deformation [13, 20] $\mu \to t\mu$, most of this field remains to be investigated.

3 Integrable deformations of qc-mappings and the universal Whitham hierarchy

Let us consider the $\bar{\partial}$-equation

$$S_{\bar{z}} = W\Big(z, \bar{z}, S_z\Big), \tag{5}$$

where the function W has a compact support in $\mathbb{C}$. The infinitesimal symmetries $S \to S + \epsilon f$ of (5) satisfy the Beltrami equation

$$f_{\bar{z}} = W'\Big(z, \bar{z}, S_z\Big) f_z, \tag{6}$$

where $W'(z, \bar{z}, \xi) := W_\xi(z, \bar{z}, \xi)$. By assuming that the hypothesis of Theorem 1 are fulfilled, there is a unique solution p of (6) on $\mathbb{C}$ (the basic homemorphism) such that

$$p = z + \frac{a_1}{z} + \frac{a_2}{z} + \dots, \quad z \to \infty, \tag{7}$$

and all other solutions of (6) are of the form

$$f = \Phi(p), \tag{8}$$

with Φ being an analytic function . This means that the set of infinitesimal symmetries of (5) form an infinite-dimensional ring.

Suppose we take a finite set of points $\{z_i\}_{i=1}^N$ outside the support of the function W in the complex plane, with $z_1 = \infty$. We may consider solutions of (5) $S(z, \bar{z}, \boldsymbol{t})$ depending on an infinite set of time parameters $\boldsymbol{t} := (\boldsymbol{t}_1, \dots, \boldsymbol{t}_N)$, $\boldsymbol{t}_n := (t_{1,1}, t_{1,2}, \dots)$, such that near the points $\{z_i\}_{i=1}^N$ have a prescribed behaviour

$$S(z, \bar{z}, \boldsymbol{t}) = S_i(z, \boldsymbol{t}_i) + \text{holomorphic part}, \quad z \to z_i, \tag{9}$$

where the functions S_i are singular at z_i. Thus for every time t_A, $(A = (i, n))$ we have a symmetry of (5)

$$f_A = \frac{\partial S}{\partial t_A}, \tag{10}$$

and from Theorem 1 it follows that for any appropriate solution $S(z, \bar{z}, \boldsymbol{t})$ of (5) there exist a unique family of basic homeomorphisms on $\mathbb{C}$

$$p = z + \frac{a_1(\boldsymbol{t})}{z} + \frac{a_2(\boldsymbol{t})}{z} + \dots, \quad z \to \infty.$$

We assume that p can be characterized as $p(z, \bar{z}, \boldsymbol{t}) := \frac{\partial S}{\partial t_{A_0}}$ for a certain time parameter t_{A_0}. In this way, for any time t_A there exists a function $\Omega_A(p, \boldsymbol{t})$ such that

$$\frac{\partial S}{\partial t_A} = \Omega_A(p, \boldsymbol{t}). \tag{11}$$

If $A = (i, n)$ then $\frac{\partial S}{\partial t_A}$ is a solution of (6) on $G = \mathbb{C}^* - \{z_i\}$, so that $\Omega_A(p, \boldsymbol{t})$ is an analytic function of p on the image of G under $p(z, \bar{z}, \boldsymbol{t})$.

The system (11) is an infinite set of Hamilton-Jacobi type equations which characterizes the infinite family of deformations $p(z, \bar{z}, \boldsymbol{t})$ of qc-mappings. By construction, all the equations (11) are compatible, so that we have

$$\frac{\partial \Omega_A}{\partial t_B} - \frac{\partial \Omega_B}{\partial t_A} + \{\Omega_A, \Omega_B\} = 0, \tag{12}$$

where the Poisson bracket $\{,\}$ is given by

$$\{F, G\} = \frac{\partial F}{\partial p}\frac{\partial G}{\partial x} - \frac{\partial F}{\partial x}\frac{\partial G}{\partial p}. \tag{13}$$

Equations (12) constitute examples of the so-called universal Whitham hierarchy introduced in [4]. Particular choices of the functions S_i in (9) give rise to different dispersionless hierarchies.

For instance, if we take a unique reference point $z_1 = \infty$ and assume

$$S(z, \boldsymbol{t}) = \sum_{n \geq 1} z^n t_n + O(\frac{1}{z}), \quad z \to \infty, \tag{14}$$

we get the dKP hierarchy. In this case

$$p = \frac{\partial S}{\partial t_1} = z + O(\frac{1}{z}), \quad z \to \infty. \tag{15}$$

Moreover, as the functions $\frac{\partial S}{\partial t_i}$ are solutions of (6) on $\mathbb{C}$, their corresponding representations $\Omega_i(p, \boldsymbol{t})$ are entire functions of p. Therefore, according to the asymptotic behaviours (14) and (15) it is clear that $\Omega_i(p, \boldsymbol{t}) = (z^i)_+$, where $z := z(p, \boldsymbol{t})$ denotes the Laurent series obtained by eliminating z as a function of p in (15), and $(z^i)_+$ is the part corresponding to the nonnegative powers of p in the expansion of z^i.. Hence, we conclude that

$$\frac{\partial S}{\partial t_i} = (z^i)_+, \quad i \geq 2. \tag{16}$$

The first two equations read

$$\begin{aligned} \frac{\partial S}{\partial t_2} &= \Big(\frac{\partial S}{\partial t_1}\Big)^2 + u(\boldsymbol{t}), \\ \frac{\partial S}{\partial t_3} &= \Big(\frac{\partial S}{\partial t_1}\Big)^3 + \frac{3}{2}u(\boldsymbol{t})\,\frac{\partial S}{\partial t_1} + \frac{3}{4}\,v(\boldsymbol{t}), \end{aligned} \tag{17}$$

where $v_{t_1} = u_{t_2}$ and the function u obeys the dKP equation

$$\Big(u_{t_3} - \frac{3}{2} u\, u_{t_1}\Big)_{t_1} = \frac{3}{4} u_{t_2 t_2}. \tag{18}$$

It must be noticed that by eliminating u and v in (17) we get an autonomous partial differential equation for S

$$\frac{\partial^2 S}{\partial t_1 \partial t_3} = \frac{3}{4} \frac{\partial^2 S}{\partial t_2^2} + \frac{3}{2} \Big[\frac{\partial S}{\partial t_2} - \Big(\frac{\partial S}{\partial t_1}\Big)^2\Big] \frac{\partial^2 S}{\partial t_1^2}. \tag{19}$$

It is just the first member of a hierarchy of autonomous equations for S which can be derived from (16).

4 Examples of qc-mappings and their dKP deformations

To illustrate our analysis let us consider the dKP hierarchy and a $\bar{\partial}$-problem of the form

$$S_{\bar{z}} = \theta(1 - z\bar{z}) W_0\Big(S_z\Big), \tag{20}$$

where $\theta(\xi)$ is the usual Heaviside function and $W_0(\xi)$ is an arbitrary differentiable function. Observe that (20) implies

$$m_{\bar{z}} = W_0'(m) m_z, \quad m := S_z, \quad |z| < 1,$$

where $W_0' = \frac{\mathrm{d}\,W_0}{\mathrm{d}\,m}$. This equation can be solved at once by applying the methods of characteristics, so that the general solution S_{in} of (20) inside the unit circle $|z| < 1$ is implicitly characterized by

$$\begin{gathered} S_{in} = W_0(m)\bar{z} + mz - f(m), \\ W_0'(m)\bar{z} + z = f'(m), \end{gathered} \tag{21}$$

where $f = f(m)$ is an arbitrary differentiable function. Notice that according to the second equation in (21), we have

$$f'(m_0) = z, \quad m_0 := m(z, \bar{z})|_{\bar{z}=0},$$

so that $f'(m_0)$ is the inverse function of $m_0 = m_0(z)$.

The solution S_{out} of (20) outside the unit circle is any arbitrary analytic function ($\bar{z}$-independent). However, in order to obtain a generalized solution of (20) with locally square-integrable generalized derivatives we impose the continuity of S at the unit circle

$$S_{out}(z) = S_{in}(z, \frac{1}{z}), \quad |z| = 1. \tag{22}$$

Moreover, as we are dealing with the dKP hierarchy, we require S_{out} to be of the form

$$S_{out} = \sum_{n\geq 1} z^n T_n + \sum_{n\geq 1} \frac{s_n(\boldsymbol{T})}{z^n}. \tag{23}$$

In summary we may proceed as follows. Firstly we take a function $m_0 = m_0(z, \boldsymbol{a})$ depending on z and a certain set of undetermined parameters $\boldsymbol{a} = (a_1, \ldots, a_n)$, then we get $f = f(m, \boldsymbol{a})$ and solve for $m = m(z, \bar{z}, \boldsymbol{a})$ in the second equation of (21). The functions $S_{in}(z, \bar{z}, \boldsymbol{a})$ and $S_{out}(z, \boldsymbol{a})$ are determined by means of the first equation of (21) and (22), respectively. Finally, we impose S_{out} to admit an asymptotic expansion of the form (23) and find the parameters $\boldsymbol{a}$ as functions of the dKP times $\boldsymbol{t}$. According to the results described in Section 3, if we get solutions S of (20) with enough regular generalized derivatives S_z and $S_{\bar{z}}$, they will lead to solutions of the dKP hierarchy which, for suitable values of the parameters $\boldsymbol{t}$, will determine deformations of qc-mappings. We observe that the crucial property ensuring that a function S of the form (23) leads to a solution of the dKP hierarchy is that all the derivatives $\frac{\partial S_{in}}{\partial t_i}$ can be written as polynomials in $\frac{\partial S_{in}}{\partial t_1}$. In this sense it is helpful to notice that in our method S depends on $\boldsymbol{t}$ through the functions $\boldsymbol{a} = \boldsymbol{a}(\boldsymbol{t})$ so that from (21)

$$\frac{\partial S_{in}}{\partial t_i} = -\frac{\partial f(m, \boldsymbol{a}(\boldsymbol{t}))}{\partial t_i}.$$

Therefore, a way to verify that S leads to a dKP solution is to check that the derivatives $\frac{\partial f(m,\boldsymbol{a}(\boldsymbol{t}))}{\partial t_i}$ can be expressed as polynomials in $\frac{\partial f(m,\boldsymbol{a}(\boldsymbol{t}))}{\partial t_1}$.

Let us consider a couple of examples with $W_0(m) = m^2$. In this case Equations (20) become

$$S_{in} = m^2\bar{z} + mz - f(m), \quad 2m\bar{z} + z = f'(m). \tag{24}$$

Example 1. If we take $m_0 = \frac{1}{a}(z-b)$,then $f(m) = \frac{a}{2}m^2 + bm + c$ and $m = \frac{z-b}{a-2\bar{z}}$. Thus we get

$$S = \begin{cases} \frac{1}{2}\frac{(z-b)^2}{a-2\bar{z}} - c, & |z| \leq 1 \\ \\ \frac{1}{2}\frac{z(z-b)^2}{az-2} - c, & |z| \geq 1. \end{cases}$$

Notice that the regularity of S inside the unit circle requires

$$|a| > 2, \tag{25}$$

which is in agreement with the required analyticity of S on $|z| > 1$.

Now we have

$$S = \frac{1}{2a}z^2 + (\frac{1}{a^2} - \frac{b}{a})z + \frac{2}{a^3} + \frac{b^2}{2a} - \frac{2b}{a^2} - c + O(\frac{1}{z}), \quad z \to \infty,$$

so that in order to fit with the dKP hierarchy we have to identify

$$a = \frac{1}{2t_2}, \quad b = 2t_2 - \frac{t_1}{2t_2}, \quad c = \frac{2}{a^3} + \frac{b^2}{2a} - \frac{2b}{a^2}.$$

Notice that

$$\frac{\partial f}{\partial t_i} = \frac{\partial a}{\partial t_i}m^2 + \frac{\partial b}{\partial t_i}m + \frac{\partial c}{\partial t_i}.$$

Hence, as a is t_1-independent we have that $\frac{\partial f}{\partial t_2}$ can be written as a quadratic polynomial in $\frac{\partial f}{\partial t_1}$.

Example 2. If we take $m_0 = az^2 + bz + c$, then from (24) we get

$$\begin{aligned} f(m) &= -\frac{b}{2a}m + \frac{1}{12a^2}(4am + b^2 - 4ac)^{\frac{3}{2}} + d \\ &= -\frac{b}{2a}m + \frac{1}{12a^2}(4a\bar{z}m + 2az + b)^3 + d, \end{aligned} \tag{26}$$

and

$$m = \frac{1}{8\bar{z}^2}\Big(\frac{1}{a} - 4(z + \frac{b}{2a})\bar{z} - \sqrt{\frac{4}{a}(\frac{b^2}{a} - 4c)\bar{z}^2 - \frac{8}{a}(z + \frac{b}{2a})\bar{z} + \frac{1}{a^2}}\ \Big). \tag{27}$$

Hence we have

$$S_{in} = (z + \frac{b}{2a})m + \bar{z}m^2 - \frac{1}{12a^2}(4a\bar{z}m + 2az + b)^3 - d. \tag{28}$$

It is clear that in order to ensure the regularity of S_{in} on $|z| \geq 1$ and the analyticity of S_{out} on $|z| > 1$ we have to require

$$\frac{4}{a}(\frac{b^2}{a} - 4c)\bar{z}^2 - \frac{8}{a}(z + \frac{b}{2a})\bar{z} + \frac{1}{a^2} \neq 0, \quad |z| < 1,$$

$$\frac{4}{a}(\frac{b^2}{a} - 4c)\frac{1}{z^2} - \frac{8}{a}(z + \frac{b}{2a})\frac{1}{z} + \frac{1}{a^2} \neq 0, \quad |z| \geq 1.$$

One can see that this can be achieved provided

$$\frac{|B|^2}{64} > (|A| + |B| + |C|), \quad (\frac{|B|^2}{64} - |B| - |C|)^2 > 4|A|(\frac{|B|^2}{64} - |B|),$$

where

$$A = \frac{4}{a}(\frac{b^2}{a} - 4c), \quad B = \frac{8}{a}, \quad C = -4\frac{b}{a^2}.$$

Notice also that S_{in} is regular at the origin as

$$\lim_{z\to 0} m = \frac{1}{2} - 8\frac{C^2}{B^3}.$$

In order to introduce the dKP times in our solution it is helpful to use the following identity valid on the unit circle $\bar{z} = z^{-1}$

$$S_{out,z} = S_{in,z} - \frac{1}{z^2}S_{in,\bar{z}} = m - \left(\frac{m}{z}\right)^2. \tag{29}$$

Thus from the expansion

$$m(z, z^{-1}) = -\left(\frac{1}{2} + \frac{1}{8a}(1 - \sqrt{1-8a})\right)z^2 +$$

$$\frac{b}{4a}\left(-1 + \frac{1}{\sqrt{1-8a}}\right)z + \frac{-2b^2 + (-1+8a)c}{(-1+8a)\sqrt{1-8a}} + \dots, \quad z \to \infty,$$

and by setting $S_{out,z} = 3t_3z^2 + 2t_2z + t_1 + \dots$ in (29) we get

$$a = \frac{1 + 36\,t_3 - \sqrt{1 - 36\,t_3 + 432\,t_3^2 - 1728\,t_3^3}}{2\,\left(9 + 72\,t_3 + 144\,t_3^2\right)},$$

$$b = \frac{-32\,\sqrt{1 - 8\,a}\,a^2 t_2}{\left(-1 + \sqrt{1 - 8\,a}\right)\,\left(-1 + \sqrt{1 - 8\,a} + 8\,a\right)},$$

$$\begin{gathered} c = 32\,\sqrt{1 - 8\,a}\,[\left(a - 8\,a^2\right)\,\left(1 - \sqrt{1 - 8\,a} + 4\,\left(-2 + \sqrt{1 - 8\,a}\right)\,a + 8a^2\right)\,t_1 + \\ 16\,a^3\,\left(1 - \sqrt{1 - 8\,a} + 4\left(-2 + \sqrt{1 - 8\,a}\right)\,a\right)\,t_2^2] \times \\ [\left(-1 + \sqrt{1 - 8\,a}\right)^2\,\left(-1 + \sqrt{1 - 8\,a} + 8\,a\right)^3]^{-1} \end{gathered}$$

Observe that a and b are t_1-independent so that

$$\frac{\partial f}{\partial t_1} = -\frac{1}{2a}\sqrt{4am + b^2 - 4ac}\,\frac{\partial c}{\partial t_1} + \frac{\partial d}{\partial t_1}.$$

Hence it follows at once that the derivatives of f with respect to t_2 and t_3 can be written as polynomials in $\frac{\partial f}{\partial t_1}$.

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